number; C_c is a semi-empirical constant; C_x, C_{xp}, C_{cf} are the frontal, profile, and friction drag coefficients. The subscript T is for turbulence.

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HELICAL WAVES IN A LIQUID FILM ON A ROTATING DISK

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The stability of steady-state axisymmetric flow against non-axisymmetric perturbations is considered.

A number of experimental and theoretical studies have been made on wave generation in a liquid film moving on the surface of a rotating flat disk [1-7]. The stability of the limiting stationary solution for a relatively thin film was studied in [2, 3, 6, 7], using an asymptotic method [2, 3, 7] or a numerical method [6]. In this paper we study the stability of the main flow, for which the Ekman number [2] is finite; we can point out that interest in such modes stems from the desire to increase the productivity of technological processes employing the given form of film flow.

Suppose that a viscous incompressible liquid is fed at a constant flow rate Q near the center of a rotating disk. The flow of the film formed on the disk is described by the functions

$$u=\frac{u_r}{\Omega r\delta^2},\ v=\frac{1}{\delta^2}\left(\frac{u_\theta}{\Omega r}-1\right),\ w=\frac{u_z}{\Omega H_c\delta^2},\ p=\frac{p_f}{\rho\Omega^2H_c^2},\ h=\frac{h_f}{H_c},$$

where $\delta = H_c\sqrt{\Omega/\nu}$, and δ^{-2} is the Ekman number. As independent variables we use the quantities $x = \ln(r/R)$, θ, $y = z/H_c$ and $s = \Omega t/\delta^2$. The system of equations and boundary conditions for determining u, v, w, p, and h is given in [3].

The hypothesis of local plane-parallelism [3] is used to study the stability of steady-state axisymmetric flow $U_1(x, y)$, $V_1(x, y)$, and $H_1(x)$, obtained numerically [8], in which the initial thickness of the film at r = R is used as the characteristic quantity H_c; the nonstationary solution in this case is written as [6]

$$u(x,\ \theta,\ y,\ s)=U_1(x_0,\ y)+H_0^2u_1(\xi,\ \theta,\ \eta,\ \tau),\ \ v=V_1+H_0^2v_1,$$
$$w=\frac{H_0^2}{\varepsilon_0}w_1,\ p=\frac{H_0}{\varepsilon_0}p_1,\ h=H_1+H_0h_1, \tag{1}$$

where $x_0 = \ln(r_0/R)$, $H_0 = H_1(x_0)$, $\varepsilon_0 = \varepsilon(x_0)$, $\xi = (x - x_0)/(\varepsilon_0 H_0)$, $\eta = y/H_0$, and $\tau = s/H_0^2$. Substituing solution (1) into the initial equations and boundary conditions and linearizing, we obtain a problem for small perturbations, which have wave solutions of the form $f_1(\xi,\theta,\eta,\tau)=f_2(\eta)\exp i(\alpha\xi+n\theta-\omega\tau_1)$, where $\tau_1 = \mathrm{Re}\tau$, $\mathrm{Re} = U_* H_*/\nu$, $H_* = H_c H_0$, and $U_* = r_0\Omega^2H_*^2/\nu$. After some manipulations, we can obtain the following problem for amplitude functions:

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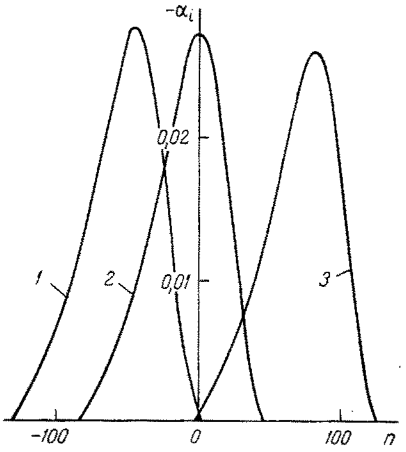

Fig. 1. Gains at $x_0 = 2$: 1) $\omega = 0.01$, 2) 0.12, 3) 0.32.

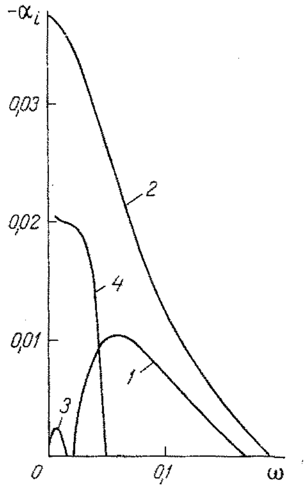

Fig. 2. Gains at $x_0 = 0.2$: 1) $n = 0$. 2) -31, 3) -1. 4) -10.

$$w^{\mathrm{IV}} - \left\{\frac{2\alpha^2 B_2}{B_1}\left[1 - \frac{i\varepsilon_*}{\alpha B_2}\left(2 - \frac{4i\varepsilon_*}{\alpha} + 3\varepsilon_*^2\beta^2\right)\right] + \right.$$

$$\left. + \mathrm{Re}\,(i\alpha E + 2\varepsilon_* U)\right\} w'' - \frac{2\varepsilon_*^2\beta\,\mathrm{Re}}{B_1}(Vw')' + \alpha\,[\alpha^2 B_2(\alpha B_3 + i\,\mathrm{Re}\,E) +$$

$$+ i\,\mathrm{Re}\,(B_1 U'' + \beta\varepsilon_* V'')]\,w + 2i\alpha\varepsilon_*\left(B_1 + \frac{\varepsilon_*^2\beta^2}{B_1}\right)[(i\alpha\beta\varepsilon_* - V\,\mathrm{Re})v' - V'\mathrm{Re}\,v] = 0,$$

$$v'' - \left[\alpha^2 B_3 + i\alpha\,\mathrm{Re}\,E + 2\varepsilon_*(U\,\mathrm{Re} - i\alpha) + \frac{4i\beta\varepsilon_*^3}{\alpha B_2}(i\alpha\beta\varepsilon_* - V\,\mathrm{Re})\right] v -$$

$$- \frac{i\beta\varepsilon_*}{\alpha B_2} w''' + \frac{i\varepsilon_*}{\alpha B_2}[i\alpha\beta\,\mathrm{Re}\,E + \alpha^2\beta B_3 + 2\,\mathrm{Re}\,(\varepsilon_*\beta U - V)]\,w' +$$

$$+ \frac{B_1\,\mathrm{Re}}{B_2}(\varepsilon_*\beta U' - V')\,w = 0, \tag{2}$$

$$\eta = 0: w = w' = v = 0,$$

$$\eta = 1: w'' + B_1\left[\alpha^2 - \frac{1}{E}(U'' - 2i\alpha\varepsilon_* U)\right] w + i\alpha\beta\varepsilon_* v' = 0,$$

$$v' + i\left[\alpha\beta\varepsilon_* - \frac{1}{\alpha E}(V'' - 2i\alpha\beta\varepsilon_*^2 U)\right] w = 0,$$

$$v''' - \left\{\frac{3\alpha^2}{B_1}\left[B_3 - \frac{4i\varepsilon_*}{\alpha}\left(1 - \frac{i\varepsilon_*}{\alpha} + \frac{2\varepsilon_*^2\beta^2}{3}\right)\right] + \right.$$

$$\left. + \mathrm{Re}\left(i\alpha E + 2\varepsilon_* U + \frac{2\beta\varepsilon_*^2}{B_1} V\right)\right\} w' + \frac{i\alpha^3\,\mathrm{Re}\,B_2 B_3}{\mathrm{We}\,E} w +$$

$$+ 2i\alpha\varepsilon_*\left(B_1 + \frac{\varepsilon_*^2\beta^2}{B_1}\right)(i\alpha\beta\varepsilon_* - V\,\mathrm{Re})\,v = 0,$$

where $\varepsilon_* = \varepsilon_0 H_0$, $\mathrm{We} = \rho U_*^2 H_*/\sigma$, $U = U_1/H_0^0$, $V = V_1/H_0^2 + D^{-2}$, $D = \delta H_0$, $E = U - c + \varepsilon_*\beta V$, $B_1 = 1 - 2i\varepsilon_*/\alpha$, $B_2 = B_1 + (\varepsilon_*\beta)^2$, $B_3 = 1 + (\varepsilon_*\beta)^2$, $\beta = n/\alpha$ and $c = \omega/\alpha$. A prime in Eqs. (2) denotes differentiattion with respect to the variable η and the subscript of the functions v_2 and w_2 has been omitted. The velocity components of the main flow depend on x_0 as parameter. We note that in [2, 6] problem (2) is considered without terms of the order of $O(\varepsilon_*, D^4)$, which corresponds to relatively thin films.

TABLE 1. Eigenvalues of Perturbations with Largest Gains at $x_0 = 2$ and at Given Values of ω

ω	n	α_r	α_i
0,01	—44	0,153	—0,0278
0,06	—23	0,151	—0,0276
0,12	1	0,152	—0,0275
0,18	25	0,154	—0,0272
0,32	82	0,154	—0,0263

TABLE 2. Eigenvalues of Perturbations of the Second Kind at $x_0 = 0.2$, $n = 1$

ω	α_r	α_i
0,0025	0,0143	—0,00217
0,0055	0,0301	—0,00247
0,0100	0,0545	—0,00198
0,0130	0,0713	—0,00112

As the parameters of the problem for the eigenvalues of (2) we use the quantities ε_*, D and $F = \rho\sqrt{\nu^3\Omega}/\sigma$, with $Re = D^4/\varepsilon_*$, and $We = FD^7/\varepsilon_*^2$. We consider time-periodic perturbations, i.e., we calculate the complex wave numbers α for given real values of ω and integral $-n$; the flow is unstable if solutions with $\alpha_i < 0$ exist (here and below the subscript i corresponds to the imaginary part of the quantity and r corresponds to the real part). The numerical method [6] is used to solve (2).

The stability is studied on the example of the flow of a film of water over the surface of a disk rotating with angular velocity $\Omega = 8$ rps; we assume that the flow rate $Q = 15\cdot10^{-6}\cdot m^3/s$ and that the film thickness $H_c = 15\cdot10^{-5}$ m at R = 0.025 m; the indicated values correspond to the conditions of the experiments in [2]. The initial profiles of the velocity components for calculating the steady-state flow are chosen in the form

$$U_1(0,\ y) = \frac{3Q\nu}{2\pi R^2\Omega^2 H_c^3}\left(y - \frac{y^2}{2}\right),\ V_1 = -\ \frac{1}{\delta^2};$$

the number of flow lines, used in the numerical method of [8], is N = 10.

Problem (2) is solved for some values of x_0 to obtain qualitative conclusions about the shape of the waves in the flow under consideration; examples of the calculations are given in Figs. 1 and 2.

At $x_0 = 2$ ($r_0 = 0.185$ m) the main flow virtually coincides with the limiting flow [2, 3, 6, 8], with $\varepsilon_* = 0.236\cdot10^{-3}$ and $D^2 = 0.0957$. The solutions of (2), obtained by the parametric continuation of the eigenvalues from [6], show that at n = 0 (concentration perturbations) the largest gain $-\alpha_i = 0.0274$ is reached at $\omega = 0.12$ (to within 0.01), with $\alpha_r = 0.156$ and $c_r = 0.77$. From Fig. 1 it follows that as decreases the largest value of $-\alpha_i$ shifts to the region of negative values of n (Table 1), to which there correspond perturbations that at a fixed time are helices which untwist in the direction of disk rotation. Since perturbations for which the assumption of local plane-parallelism is not valid correspond to large values of [n], such solutions are physically meaningless; e.g., at $\omega = 0.01$ the increment of the radius per turn about the axis for the most unstable perturbation at a fixed time is 0.0984 m, i.e., is comparable with the characteristic flow radius. From Table 1 it follows that the values of α_r at the "vertices" of cross sections of the surface $-\alpha_i$ by the planes ω = const are similar. From the reported results and the hypothesis that perturbations with maximum gain can occur it can be concluded that in the section $x_0 = 2$ the waves correspond to $\alpha_r = 0.15$-0.16 and some positive value of n, whose determination involves a more refined estimate of the change in the main flow along the radius.

Figure 2 shows the graphs of the gains at $x_0 = 0.2$ ($r_0 = 0.0305$ m); the main flow has the parameters $\varepsilon_* = 0.649\cdot10^{-2}$ and $D^2 = 1.97$ and a numerical velocity profile, given by the coefficients of the Chebyshev-series expansion of its components. Curves 1 and 2 correspond to perturbations obtained by the parametric continuation of the results of [6]. Analysis of the solutions, similar to that carried out for the cross section $x_0 = 2$, shows that some nonpositive value of n and $\alpha_r = 0.25$-0.26 correspond to the perturbation with the largest gain.

Shkadov [3] showed that at certain values of the parameters the problem of the stability of a film on a disk is close to the corresponding problem for a gravitational film for which approximate unstable and stable solutions exist. The parametric continuation of the unstable solution was considered above. The continuation of the second solution for the problem of a film on a disk reveals that it can pass into the unstable region; perturbations of this type correspond to curves 3 and 4 in Fig. 2; examples of eigenvalues are given in Table 2.

Numerical analysis in the cross section $x_0 = 0.2$ shows that perturbations of the second kind are unstable at $n > 0$, i.e., only helical waves untwisting in the direction opposite to the disk rotation are possible. Small values of α_r [the break in curve 4 (Fig. 2) is due to passage to $\alpha_r < 0$] lead to a more stringent limitation on the values of n than for perturbations of the first kind, because the homogeneity of the main flow is disrupted.

Waves of two types are observed in the experiments; concentric waves moving from the center of the disk [1, 2, 4, 5] and helical waves that untwist in the direction of disk rotation and are immobile with respect to the disk; these waves exist during the flow of relatively thick films [1, 2]. Perturbations of the first kind can be assigned to the concentric waves. Immobile helical waves do not appear in the local model used, which includes the hypothesis that perturbations with the largest gain occur, since if they are for them to be stationary relative to the disk they must satisfy the condition $n = \omega D^2/\varepsilon_*$, and, hence, $n > 0$, but the requirement about the direction in which these helical waves untwist determines that $n < 0$. It seems that this mode is an example of steady-state axisymmetric flow.

Waves, found by solving (2), which correspond to perturbations of the second kind are not described in the experimental studied. The difficulty encountered in observing them when they do exist stems from the fact that they correspond to rather thick films, whose steady-state flow is characterized by oscillations of the free surface [8]. We point out that helical waves which untwist in the direction opposite to the rotation of the disk can be almost immobile relative to the disk. For example, the "stationary" frequency for the flow at $x_0 = 0.2$ for $n = 1$ considered above is $\omega = n\varepsilon_*/D^2 = 0.0033$ and from Fig. 2 it follows that the value lies in the region of instability.

NOTATION

Here Ω denotes the angular velocity of the disk; ρ, ν, and σ are, respectively, the density, kinematic viscosity, and the surface tension of the liquid; t is the time, r, θ, and z is the fixed cylindrical coordinate system bound to the axis of rotation of the disk, u_r, u_θ, and u_z are the velocity components; p_f is the pressure, h_j is the film thicknees; R is the minimum radius of the flow region; r_0 is the value of the radius at which stability is studied; H_c is the characteristic film thickness, and $\varepsilon = H_c/r$.

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